Modeling Ad hoc Sensor Networks using Random Graph Theory

Haruko Kawahigashi, Yoshiaki Terashima, Naoto Miyauchi, Tetsuo Nakakawaji
Information Technology Laboratory
Mitsubishi Electric Corporation
Kamakura, 247-8501 JAPAN
haruko@isl.melco.co.jp

Abstract—Modeling of ad hoc sensor networks becomes difficult when uncertain features of the network increase. Deterministic modeling is difficult and some stochastic arguments should be introduced instead. In this paper, we introduce the concept of random networks. One remarkable feature of random graphs is that a phase transition occurs as the probability of edge connection increases. At the critical probability, fragmented pieces of edges suddenly start to be mutually connected, forming one large component. This graph-theoretical change parallels phase transitions in states of matter, e.g. the jump from water to ice. In ad hoc sensor networks, the nodes are connected by wireless links. In order to meet this requirement, we propose a model using percolation, a kind of random graph where the edges are formed only between the nearby nodes. We also present some numerical examples to simulate the jump effects of the phase transition.

Keywords- ad hoc networks; sensor networks; random graphs; percolation

I. INTRODUCTION

Sensor networks where numerous nodes are dispersed within a certain area have been discussed much in the era of ubiquitous networks. In the sensor networks, the networks are formed in an ad hoc manner, that is, the following conditions hold. No fixed infrastructure such as base station or wired link is assumed. The networks are formed adaptively. The nodes are connected by the wireless links. The data are passed through multi hop mutual relaying [1]—[3].

Modeling of ad hoc sensor networks becomes difficult when uncertain features of the network increase [4] [5]. In conventional wired networks, communication nodes are represented by graph vertices, and links are represented by edges [6]. Deterministic graphs can be drawn because their topological configurations are known. In ad hoc sensor networks, however, the wireless links are uncertain and unstable. Moreover, in some cases, the node statuses are uncertain and unstable as well. An example is a sensor network where the small node apparatuses are dispersed from air vehicles to the land surface [1] [3]. The exact positions of the nodes are not known, nor the link connections between the nodes. Especially when the number of the nodes is large, deterministic modeling is difficult and some stochastic arguments should be introduced.

In this paper, we introduce the concept of random networks for modeling of ad hoc sensor networks. Suppose a sensor network with \( N \) nodes and the links between the nodes are added randomly [7]—[10]. The network is modeled by a graph with \( N \) vertices fixed in advance, and adding edges between random pairs with probability \( p \). We suppose all \( \binom{N}{2} \) = \( N(N-1)/2 \) edge candidates between all pairs of \( N \) vertices have equal chance \( p \) to be added. Fig. 1 shows an example of \( N = 100 \) vertices scattered randomly in an area of 10 x 10, and edges are added with probability \( p = 0.01 \). Every edge candidate is examined independently, and added with probability \( p \).

One remarkable feature of random graphs is that a phase transition occurs as \( p \) increases. At critical probability \( p \), a sudden change of the states occur. An example is the number of components connected to the largest connected pieces. When \( p \) is small, there are many separate components, but suddenly, a large component appears as \( p \) increases. This graph-theoretical change parallels phase transitions in states of matter, e.g. the jump from water to ice or to steam.

Figure 1. Random graph (N=100, p=0.01)
By watching Fig. 1, one should notice that the model is rather close to a wired network, e.g. the World Wide Web access network, since links between nodes far apart are randomly chosen. Our aim here, however, is wireless ad hoc sensor networks, where all the links are realized by wireless access. One hop link between far apart nodes beyond radio access area is not possible. In order to meet our aim, we introduce percolation, another kind of random graph model [11]—[17]. Fig. 2 shows a percolation model represented by trees in an orchard [14]. The trees are placed regularly in lattice for simpler handling. A tree is infected to epidemic. The epidemic infects adjacent trees and spread to the orchard. We describe the detail later.

The remainder of this paper is organized as follows. At first in next section, we explain the random graph theory more in detail. Then, we describe our model. We then show some numerical example of using the model.

II. RANDOM GRAPH THEORY

A. Percolation

Our aim here is to apply an idea of random graphs to ad hoc sensor networks. (Recall that a network is called a graph in mathematical graph theory.) A random graph is a graph produced by some random procedure. Usually, the nodes are fixed beforehand and the edges are produced according to some random generation rules. In the mathematically simplest case, any edge between arbitrary two vertices is generated with the same probability, and a large amount of mathematical literature is devoted to studies of such a case as in [8] [9]. This, however, is an unrealistic model for us, because we regard an edge as a wireless connection and thus near and far pairs of vertices have to be treated differently, obviously. Thus, though our problem is within a random graph theory in a wider sense of random graphs, it is rather far from the most commonly studied cases of random graphs. Instead, our situation is more similar to a problem of percolation. In a theory of percolation, we have a regular lattice on a plane or a higher dimensional space. The vertices of the lattice are those of the random graph we consider. Then each edge of a lattice is a candidate of an edge of a random graph and it becomes an edge of the random graph with fixed probability \( p \) independently.

A classical counterpart of such a model in the real world is the following example. Suppose trees are aligned regularly in a garden and a disease is transmitted to one tree to each neighboring tree at a fixed probability. Then we study when this disease is spread indefinitely to other trees. (See Fig. 2. Of course, in the real world, the number of trees is not infinite and the trees are not exactly aligned according to a lattice, but these are matters of theoretical approximation. Our aim is to study universal features beyond this type of differences in the idealization.) If \( p \) is large, most of the vertices would be connected on the random graph and if \( p \) is small, the vertices would be rather disconnected. A rigorous mathematical study of this situation is a theory of percolation. Statistical mechanics of phase transition such as vaporization is also behind such a mathematical theory.

B. Basic Definitions

We fix an origin in the lattice. Then Fig. 3 is an example of a random graph and the connected component of the random graph is marked there. We deal with this connected component of the random graph containing the origin, and study when the expected value of this number of the vertices becomes infinite. That is, let \( C_o \) be the connected component of the random graph containing the origin and denote the number of the vertices of it by \( |C_o| \). We give its expectation value \( E(|C_o|) \) as follows.

\[
E(|C_o|) = \sum_{n=1}^\infty n P(|C_o| = n). \tag{1}
\]

Obviously, this expected value is an increasing function of the probability \( p \), and it is infinite at \( p=1 \), so there must be an infimum value of \( p \) for which the expected number of the vertices in the connected component is infinite. Roughly speaking, this value of \( p \) is a threshold beyond which the
random graph is “very connected”. (This is only one criterion of one aspect of high connectedness since the entire graph could have infinitely many components even if the one containing the origin has an infinite size, but our idea is that whatever property we may consider for connectedness, we obtain essentially the same condition. This is the universal feature of percolation.) This value is denoted by \(p_T\), where \(T\) stands for Temperley. Another criterion of such high connectivity is the probability that the connected component of the origin contains infinitely many vertices, that is, the probability \(P(\mid C_0 \mid = \infty)\). This probability also clearly takes value infinity beyond a certain point of \(m_1\). This probability also clearly are integers.                       (2)

Each of the infinitely many such edges on the integer lattice probability the origin contains infinitely many vertices, that is, the connectivity is the probability that the connected component of stands for Temperley. Another criterion of such high strictly positive beyond this value.  This threshold value is probability is 0 up to some threshold value mathematical study fairly easily shows as in [13] that this a relation between the former threshold value

reason we expect this approach is useful for our study of ad hoc encounter in many different situations.  This universality is one drastic change beyond this value is a universal phenomenon we existence of such a critical value and a connected component among all the vertices becomes “almost” edge surpasses a certain critical value, then the largest connected component of the origin of the resulting random graph by \(C_0\) and its number of vertices by \(|C_0|\). These mean the range where the communication from the base station is reachable and the number of the stations within the range. We study when this number \(|C_0|\) becomes infinite in this random model. In an actual situation, the total number of the vertices is, of course, finite, but in our simple model, we have infinitely many vertices, and when the component \(C_0\) has infinitely many vertices, we think that the random graph is sufficiently connected and use such a case as a model of a wireless network with sufficiently high connectivity. By the universality mentioned at the end of the last section, we expect that this interpretation of our model is justified. We set the probability of this component \(C_0\) having infinitely many vertices to be \(\theta(p)\), since this is obviously a function of the parameter \(p\). Then it has been studied by many authors and shown as in [13] that an approximate graph of the function \(\theta(p)\) is like Fig. 4- (a). In our setting, this graph is interpreted as follows. For small values of probability \(p\), the function \(\theta(p)\) has value 0, that is, the probability that the resulting random graph is highly connected is very small. Then suddenly beyond a certain value of \(p\), \(p_H\), we have a positive value of \(\theta(p)\) and this function increases quickly. This means that after the critical value \(p_H\), our random network quickly becomes highly connected, while the parameter \(p\) increases.

![Figure 4. Probability and expectation \(\chi(p)\)](image)

(a) Probability \(\theta(p)\)

(b) Expectation \(\chi(p)\)

We will apply this result to our model of an ad hoc network.

III. MODELS

A. Setting

Now we explain our model. Possible nodes of the random network are integer lattice points on the plane. That is, the vertex set is

\[\mathbb{Z}^2 = \{(m, n) \mid m, n \text{ are integers}\}.\]

The actual nodes of a network are not so aligned regularly, but this is a kind of quantization and we consider a simple model for theoretical handling. Each vertex of this graph \((m, n)\) has four candidates of edges connected to its four neighboring vertices, \((m+1, n), (m-1, n), (m, n+1), (m, n-1)\). Each of the infinitely many such edges on the integer lattice becomes an edge of the random graph with equal probability \(p\) independently. This represents that a wireless station has a communication with a neighboring station. Then we fix some base station and study connectivity of the connection network from it. We may and do take this base vertex to be the origin \((0, 0)\).

B. Probability having an Infinite Component

As in the previous section, we denote the connected component of the origin of the resulting random graph by \(C_0\) and its number of vertices by \(|C_0|\). These mean the range where the communication from the base station is reachable and the number of the stations within the range. We study when this number \(|C_0|\) becomes infinite in this random model. In an actual situation, the total number of the vertices is, of course, finite, but in our simple model, we have infinitely many vertices, and when the component \(C_0\) has infinitely many vertices, we think that the random graph is sufficiently connected and use such a case as a model of a wireless network with sufficiently high connectivity. By the universality mentioned at the end of the last section, we expect that this interpretation of our model is justified. We set the probability of this component \(C_0\) having infinitely many vertices to be \(\theta(p)\), since this is obviously a function of the parameter \(p\). Then it has been studied by many authors and shown as in [13] that an approximate graph of the function \(\theta(p)\) is like Fig. 4- (a). In our setting, this graph is interpreted as follows. For small values of probability \(p\), the function \(\theta(p)\) has value 0, that is, the probability that the resulting random graph is highly connected is very small. Then suddenly beyond a certain value of \(p\), \(p_H\), we have a positive value of \(\theta(p)\) and this function increases quickly. This means that after the critical value \(p_H\), our random network quickly becomes highly connected, while the parameter \(p\) increases.

C. Expectation Value being Infinite

Next, we make a similar, but different consideration of connectivity of the random graph in our model. As in the previous section, we study the expectation value \(E(C_0)\) of the number \(|C_0|\) and now denote it by \(\chi(p)\) as a function of \(p\). This is also an increasing function of \(p\) clearly, and its graph is approximately as in Fig. 4-(b). Note that \(\chi(1) = \infty\), so the function \(\chi(p)\) takes value infinity beyond a certain point of \(p\). We denote this “certain” point by \(p_T\). In 1980, [12] proved that these value \(p_H\) and \(p_T\) are both equal to 1/2 for the current
model, and a further generalization to a higher dimension has been made by [16] and [17]. Their proofs use a very high level of mathematical intricacy handling various inequalities, so we do not bother to explain these details here. The main point important for our model is that while $p$ increases, some drastic change of higher connectivity occurs at probability $1/2$ and we apply this idea to a concrete wireless networks.

IV. NUMERICAL EXAMPLES

In this section, we show numerical examples obtained by computer simulation.

A. Phase Transition at the Critical Probability

Fig. 5 shows how the origin-connected component evolves as the probability $p$ increases. Set an area of $10 \times 10$, and vertices are placed at the lattice points. Every edge candidate of length one is added or ON with independent and identical probability $p$. The origin is set at the center of the area. Fig. 5-(a) shows a case when $p = 0.1$. Small fragments of the edge components are scattered in the area. Fig. 5-(b) shows a case when $p = 0.3$. Some edge components start to grow, but the most of ON edges are still fragmented. The largest component grows when $p$ increases $0.5$ to $0.7$ (Fig. 5-(c) (d)), and all the ON edges are connected when $p = 0.9$ (Fig. 5-(e)).

In order to examine occurrence of the phase transition, we simulated percolation of one hundred cases. Fig. 6 shows ratio of the number of ON edges to the number of edge candidates in $10 \times 10$ areas. (Though we considered the number of the vertices before, we count the number of the edges here. This is for simplicity and causes no real difference because we are interested in when the component containing the origin becomes infinite.) The ratio increases proportional to the probability $p$. Fig. 7 shows ratio of the origin-connected edges to the edge candidates. There is a gap around $p = 0.5$. In order to see this jump more clearly, we examined the cases with bigger areas of $50 \times 50$ and $100 \times 100$. Fig. 8 shows ratio of origin-connected edges to total ON edges. As the area becomes larger, the jump around $p = 0.5$ is clearer.

B. Radio Transmission

Table 1 shows IEEE802.15 Personal Area Network, as examples of transmission methods that can be used for sensor networks. The specifications, namely, frequencies, transmission distances, speeds and powers that will be available in Japan are also shown. The transmission distances are several meters to 100m.

Radio wave attenuates as the distance from the transmitter increases. The attenuation is usually described by the transmission loss between the transmitter and the receiver, or the electro magnetic field strength at the receiver with a certain transmitter condition assumed. We use the electro magnetic field strength with a conventional transmitter condition of 1kWerp (effective radiation power), namely, transmission power of 1kW using a half wavelength dipole antenna [18] [19].

We examine the radio wave strength of several to 100 m range from the transmitter. The radio wave attenuation inside buildings is difficult to analyze, and we approximate it using a well known relation in case with a half wave length dipole antenna.

$$E = -20 \log d,$$  \hspace{1cm} (3)

where $E$ denotes relative electro magnetic field strength (dB) and $d$ denotes distance (m) between the transmitter and the receiver. Fig. 9 shows the relation of the above equation. The electro magnetic field strength $E$ is set to zero at distance $d = 0$.

Probability $p$ of link establishment between a transmitter and a receiver is strongly related to the electro magnetic field strength $E$, but the exact relation is not clear. We approximate that $p$ is proportional to $E$, $p = 1$ at $d = 0$ , and $p = 0$ at $d = d_0$. The following equation shows our approximation of the probability $p$.

$$p = \frac{E(d_0) + E}{E(d_0)} = \frac{1 - \frac{20}{E(d_0)} \log d}{(4)}$$

According to the above equation, we obtain the relation between $p$ and $d$.

Here, we set $d_0 = 100$ m, namely, probability $p = 0$ at $d = 100$. Using the results of Fig. 8 and Fig. 9, we obtain a relation between ratio of origin-connected edges to total ON edges and distance $d$, as shown in Fig. 10. Three curves show the results of three kinds of area, $10 \times 10$, $50 \times 50$, and $100 \times 100$. The jump effects of the phase transition shown in Fig. 8 are seen in Fig. 10 as well, by sharp decrease of the ratio of origin-connected edges when the distance $d$ is around 10 to 20 m. In fact, the jump effects are strengthened by the effect of radio wave attenuation, and the decrease of the ratio of origin-connected edges is very sharp. We need to select the distance $d$ of the nodes or the transmission power very carefully, when we design ad hoc sensor networks, so that the distance $d$ is within the effective range, below 15 m in Fig. 10 example. If $d$ is above the effective range, the nodes will not be connected to one, leaving many pieces of fragments.

<table>
<thead>
<tr>
<th>Method</th>
<th>Bluetooth</th>
<th>ZigBee</th>
<th>UWB</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard</td>
<td>IEEE802.15.1</td>
<td>IEEE802.15.4</td>
<td>IEEE802.15.3a</td>
</tr>
<tr>
<td>Frequency</td>
<td>2.4GHz</td>
<td>2.4GHz, 868MHz</td>
<td>3.1GHz—10.6GHz</td>
</tr>
<tr>
<td>Distance</td>
<td>10—100m</td>
<td>10—75m</td>
<td>4m / 10m</td>
</tr>
<tr>
<td>Speed</td>
<td>1Mbps</td>
<td>250kbps</td>
<td>480Mbps</td>
</tr>
<tr>
<td>Power</td>
<td>120 / 4 .2mW</td>
<td>&lt; 60mW</td>
<td>&lt; 100mW</td>
</tr>
</tbody>
</table>

0-7803-8785-6/05/$20.00 (C) 2005 IEEE
V. CONCLUSION

We have introduced the concept of random graph theory for modeling ad hoc sensor networks, where the nodes are connected only by wireless links. We have proposed a model using percolation, a kind of random graph where the edges are formed only between the nearby nodes.

We have presented some numerical examples to simulate the jump effects of the phase transition. We have obtained qualitative results that the jump effect of the phase transition appears sharply by synergistic effect with radio wave attenuation as the distance between the transmitter and the receiver increases. Thus the distance between the nodes should be chosen within effective range.

In order to use the proposed model for ad hoc sensor network design, we have to choose right values for the parameters such as distance \(d_0\) where the probability \(p\) of link forming equals zero.

\begin{figure}[h]
\centering
\subfloat[p=0.1]{
\includegraphics[width=0.4\textwidth]{figure5a}
}
\subfloat[p=0.3]{
\includegraphics[width=0.4\textwidth]{figure5b}
}
\subfloat[p=0.5]{
\includegraphics[width=0.4\textwidth]{figure5c}
}
\subfloat[p=0.7]{
\includegraphics[width=0.4\textwidth]{figure5d}
}
\subfloat[p=0.9]{
\includegraphics[width=0.4\textwidth]{figure5e}
}
\caption{Percolation with different probabilities}
\end{figure}

REFERENCES


Figure 6. Ratio of ON edges to edge candidates

Figure 7. Ratio of origin-connected edges to edge candidates

Figure 8. Ratio of origin-connected / total ON edges vs. distance

Figure 9. Relative electromagnetic field strength vs distance

Figure 10. Ratio of origin-connected / total ON edges vs. distance